ICERM Conference on
Computational Challenges in the Theory of Lattices

Providence, April 2018

Variations and Applications of Voronoi's algorithm

Achill Schürmann
(Universität Rostock)

(based on work with Mathieu Dutour Sikiric and Frank Vallentin)

PRELUDE
Voronoi's Algorithm
- classically -

Lattices and Quadratic Forms

Lattices and Quadratic Forms

- Every lattice basis $A \in \mathsf{GL}_n(\mathbb{R})$ of a lattice $L = A\mathbb{Z}^n$ defines a positive definite symmetric (Gram) matrix $Q = A^t A$.

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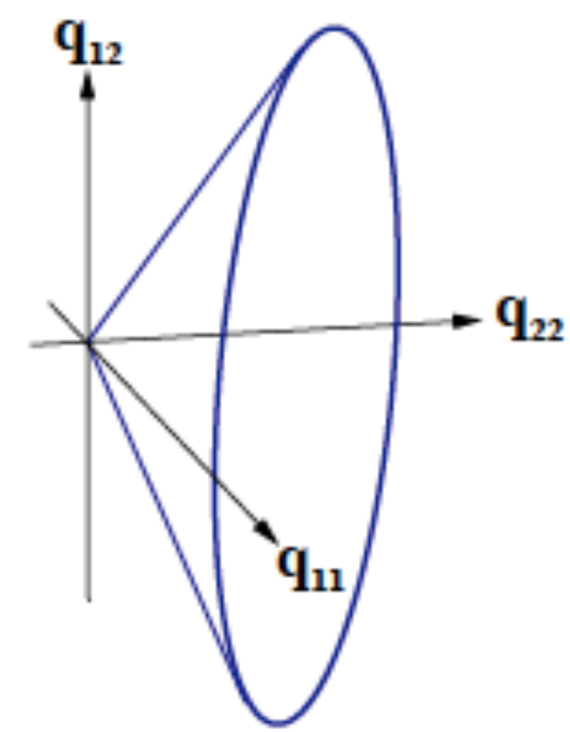


- $Q \in \mathcal{S}^n_{>0}$ defines a pos. def. quadratic form (PQF)

$$Q[x] \;=\; x^t Q x \;=\; \sum_{i=1}^{n} q_{ii} x_i^2 \;+\; 2 \sum_{i<j} q_{ij} x_i x_j$$

Different bases of a lattice yield integrally equivalent PQFs:

$$L = A\mathbb{Z}^n \quad \Leftrightarrow \quad L = (AU)\mathbb{Z}^n \text{ for } U \in \mathrm{GL}_n(\mathbb{Z})$$

$$A^t A = Q \sim Q' = U^t Q U = (AU)^t (AU)$$

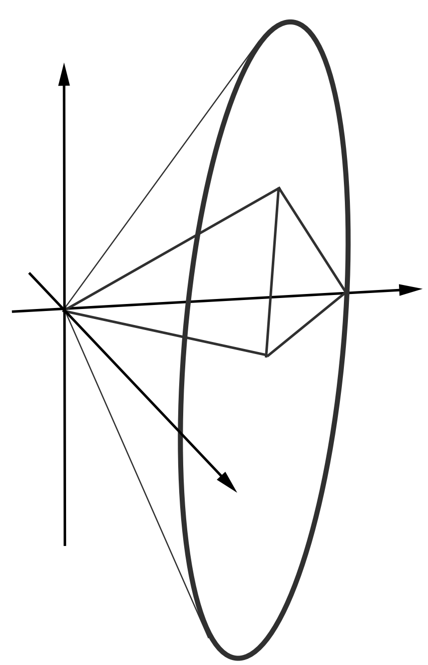
Reduction Theory

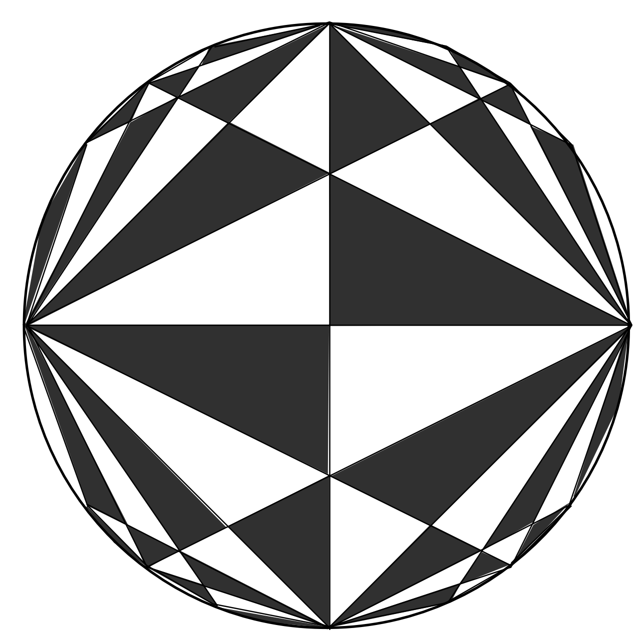
for positive definite quadratic forms

$\mathrm{GL}_n(\mathbb{Z})$ acts on $\mathcal{S}^n_{>0}$ by $Q \mapsto U^t Q U$

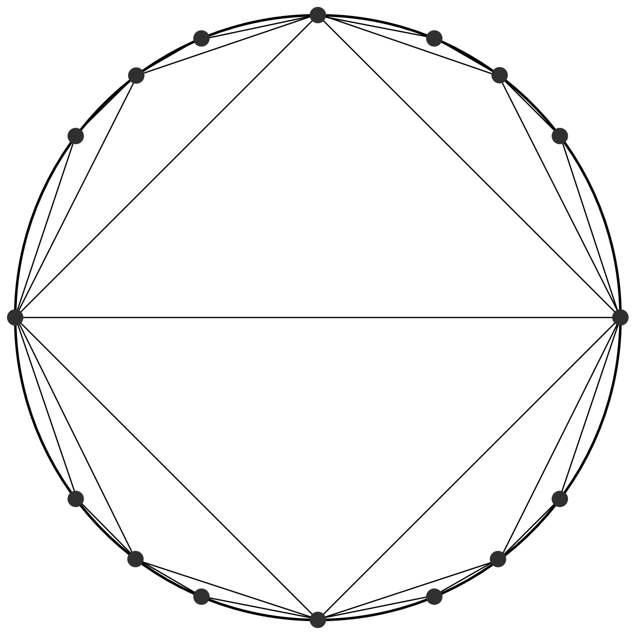
Task of a reduction theory is to provide a fundamental domain

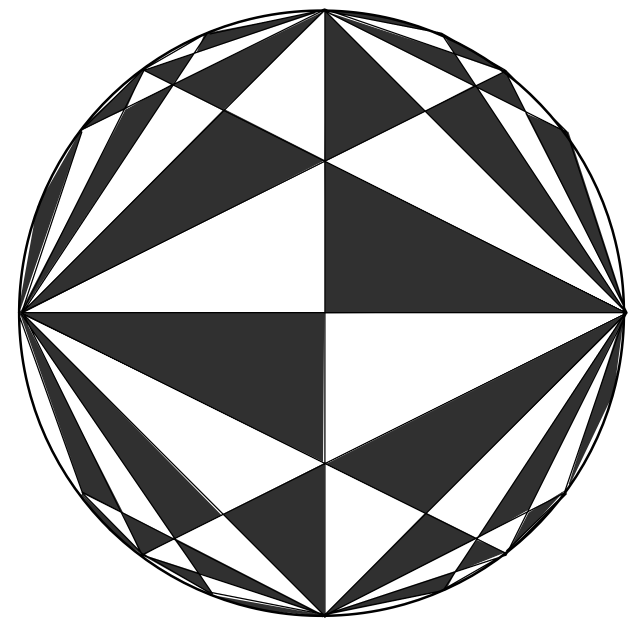
Classical reductions were obtained by Lagrange, Gauß, Korkin and Zolotareff, Minkowski and others… All the same for $n = 2$:





Voronoi's reduction idea





Georgy Voronoi
(1868 – 1908)

Observation: The fundamental domain can be obtained from polyhedral cones that are spanned by rank-1 forms only

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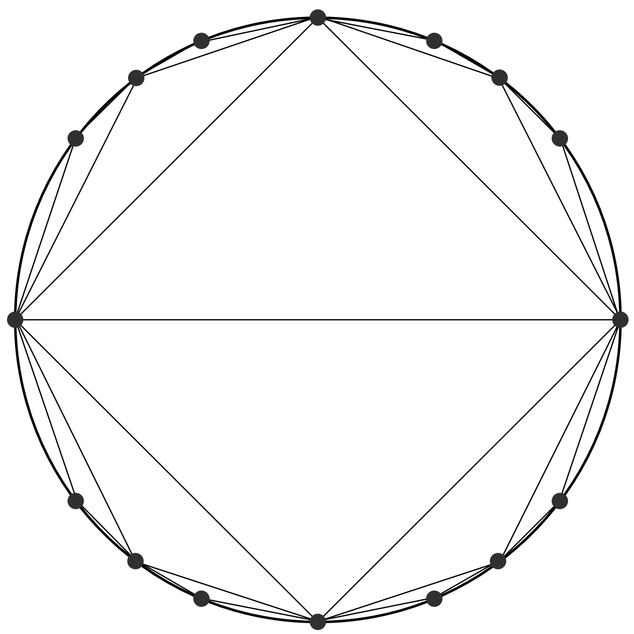

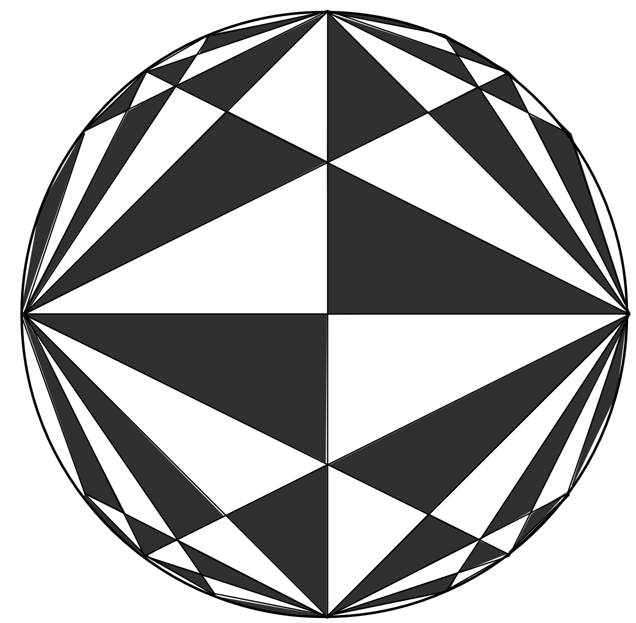

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Voronoi's algorithm gives a recipe for the construction of a complete list of such polyhedral cones up to $\mathrm{GL}_n(\mathbb{Z})$-equivalence

Perfect Forms

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(Voronoi cones are full dimensional if and only if Q is perfect!)

Ryshkov Polyhedron

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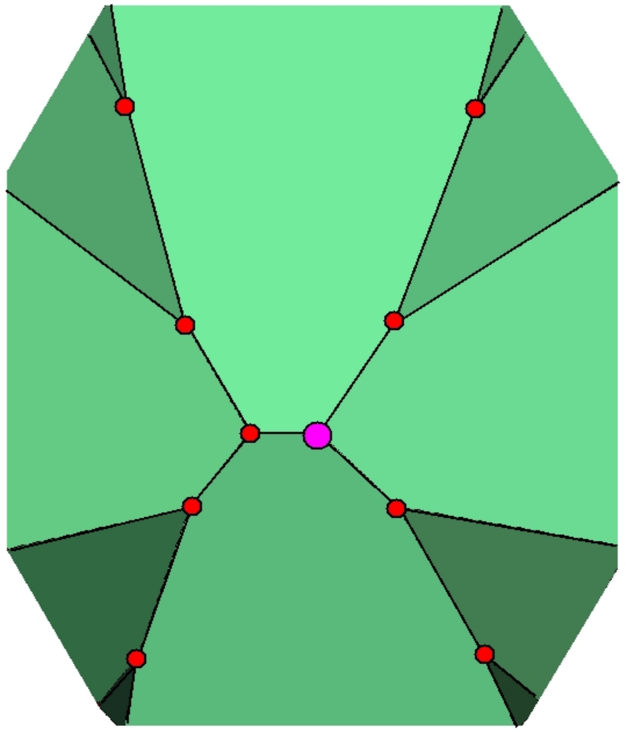

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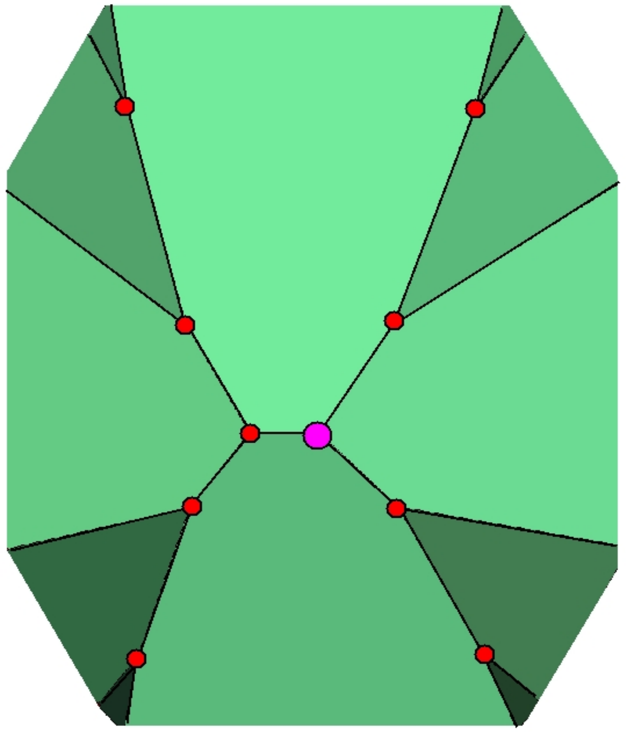
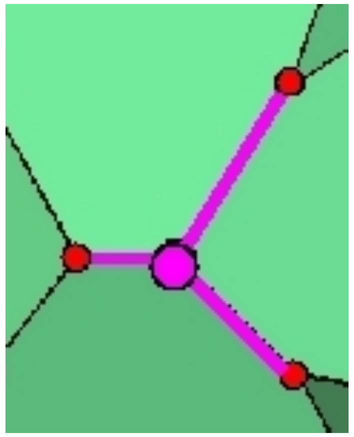
- $\mathcal{R}$ is a locally finite polyhedron
- Vertices of $\mathcal{R}$ are perfect



Voronoi's Algorithm



Start with a perfect form Q

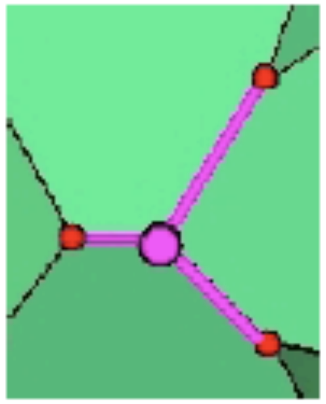
1. SVP: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\; Q' \in \mathcal{S}^n \;:\; Q'[x] \geq 1 \text{ for all } x \in \operatorname{Min} Q \;\}$$

2. PolyRepConv: Enumerate extreme rays $R_1, \ldots, R_k$ of $\mathcal{P}(Q)$
3. SVPs: Determine contiguous perfect forms $Q_i = Q + \alpha R_i$, $i = 1, \ldots, k$
4. ISOMs: Test if Q_i is arithmetically equivalent to a known form
5. Repeat steps 1.–4. for new perfect forms

Computational Results

- **BOTTLENECK**: Computing vertices of polyhedra!
- **Martinet (2003)**: "The existence of E_8 [...] makes hopeless any attempt [...] in dimension 8."



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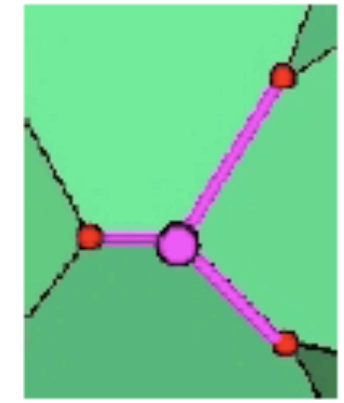

n	**# perfect forms**	**author(s)**
2	1	Lagrange, 1773
3	1	Gauß, 1840
4	2	Korkine & Zolotareff, 1877
5	3	Korkine & Zolotareff, 1877
6	7	Barnes, 1957
7	33	Jaquet-Chiffelle, 1991
8	10916	Dutour Sikirić, Sch. & Vallentin, 2007
9	> 500000	

Computer assisted proof with *Recursive Adj. Decomp. Method* (ADM) for vertex enumeration up to symmetries

(showing that the “E_8-polytope” has 25075566937584 vertices in 83092 orbits)

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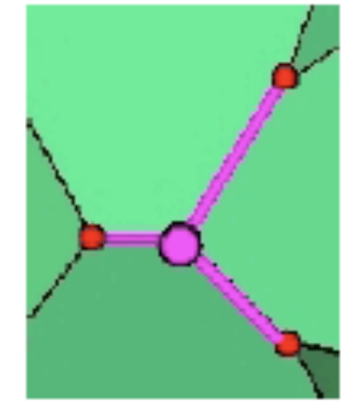

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9	$>$ 5000000	Wessel van Woerden, 2018 ?!

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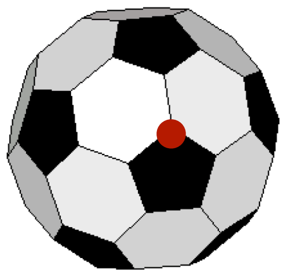

- Find initial orbit(s) / representing vertice(s)

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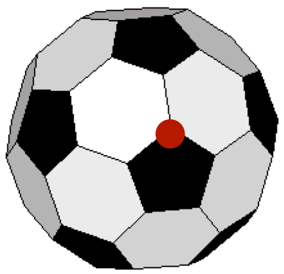

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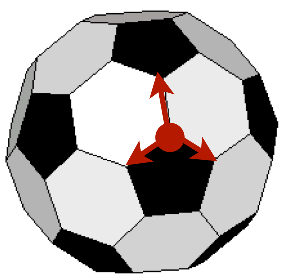

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  - enumerate neighboring vertices

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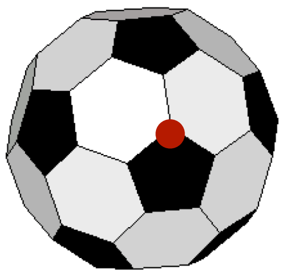

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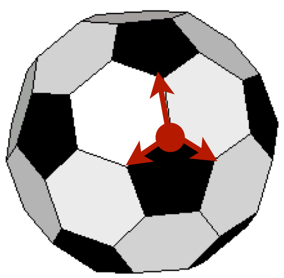

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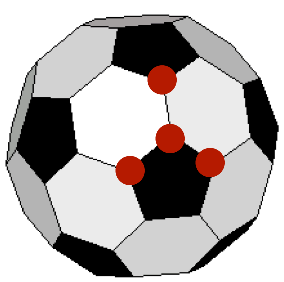

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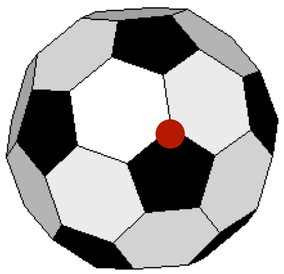

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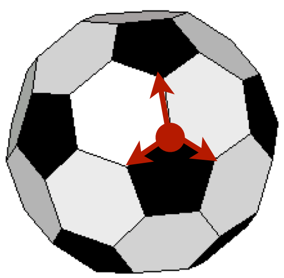

- For each new orbit representative
  - enumerate neighboring vertices (up to symmetry)
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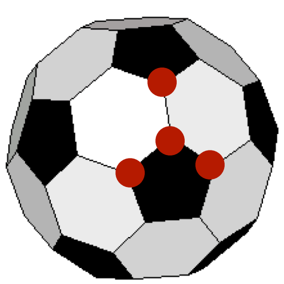

Representation conversion problem

Adjacency Decomposition Method

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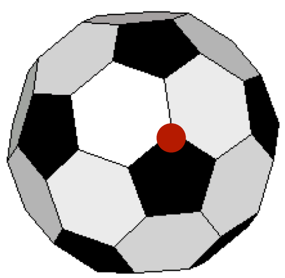

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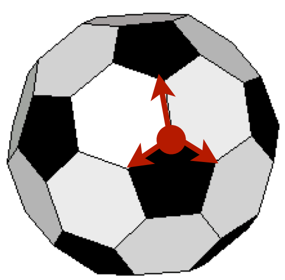

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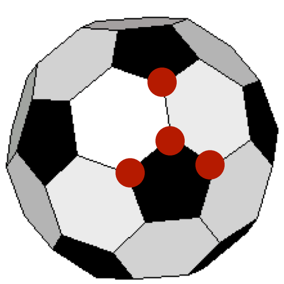

Representation conversion problem

BOTTLENECK: Stabilizer and In-Orbit computations

=> Need of efficient data structures and algorithms for permutation groups: BSGS, (partition) backtracking

Representation Conversion
in practice

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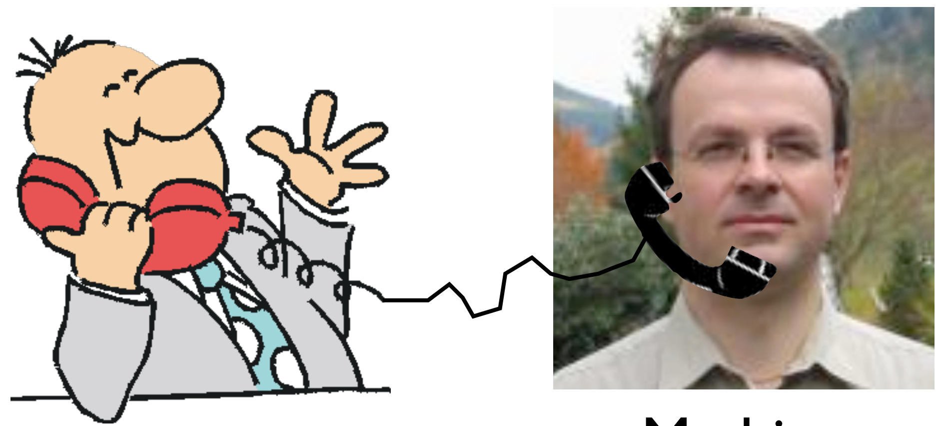
in practice

Best known Algorithm:



Mathieu

Representation Conversion
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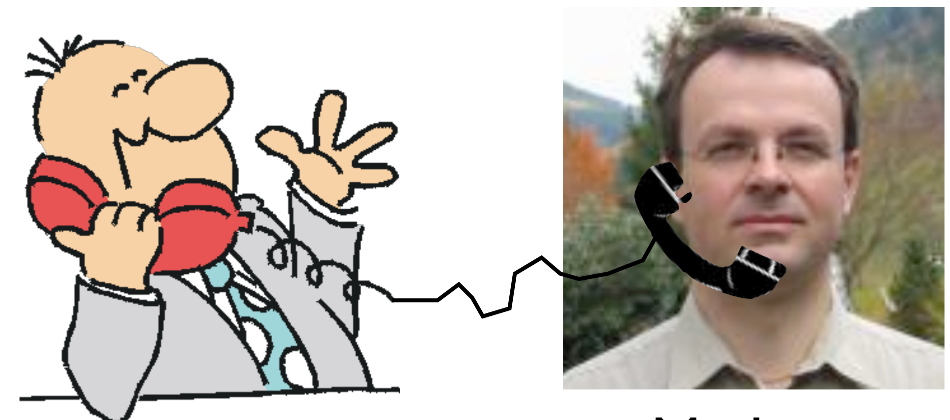


Mathieu

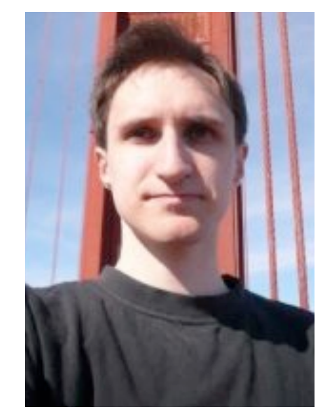
Best known Algorithm:

A C++-Tool SymPol

also available through polymake

- helps to compute linear automorphism groups
- converts polyhedral representations using

Recursive Decomposition Methods (Incidence/Adjacency)



Thomas Rehn
(Phd 2014)

http://www.geometrie.uni-rostock.de/software/

Applicaton: Lattice Sphere Packings

The lattice sphere packing problem can be phrased as:

Minimize $(\det Q)^{1/n}$ on

$$\mathcal{R} = \{ Q \in \mathcal{S}^n_{>0} : Q[x] \geq 1 \text{ for all } x \in \mathbb{Z}^n \setminus \{0\} \}$$

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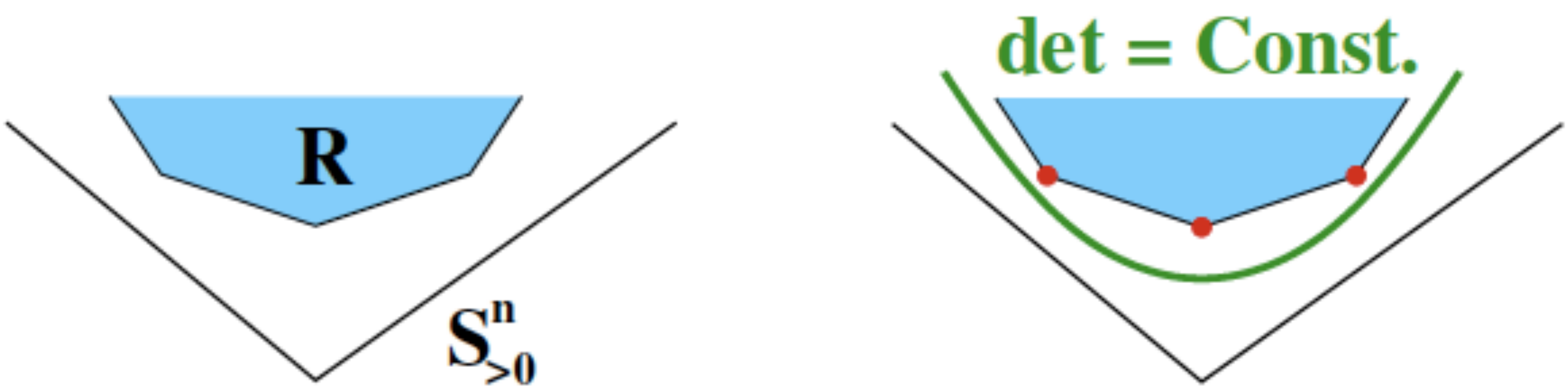

$\min_{Q \in \mathcal{R}} (\det Q)^{1/n}$ is attained at vertices of $\mathcal{R}$ (perfect forms)

Part II: Koecher's generalization and T-perfect forms

Koecher's generalization

1960/61 Max Koecher generalized Voronoi's reduction theory and proofs to a setting with a self-dual cone C

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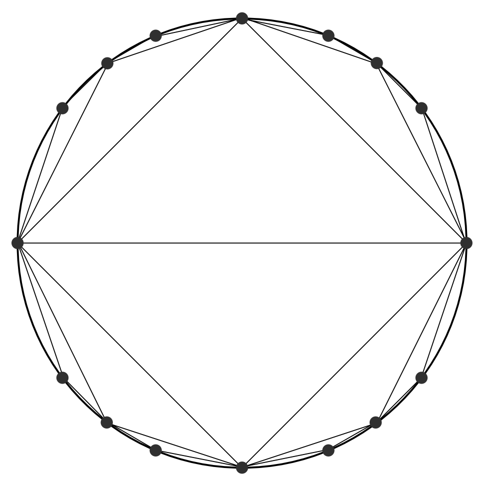

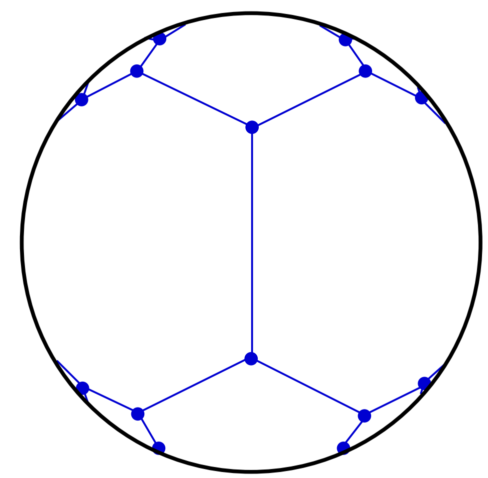

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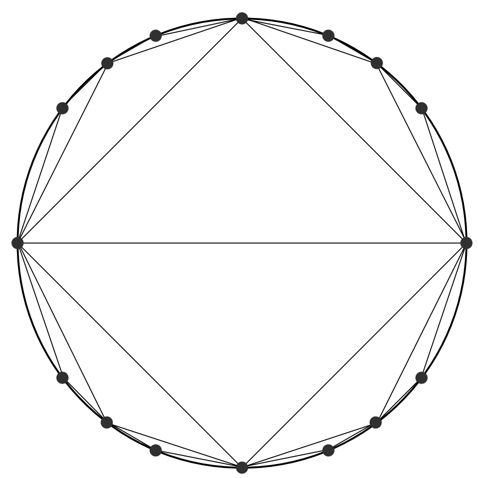

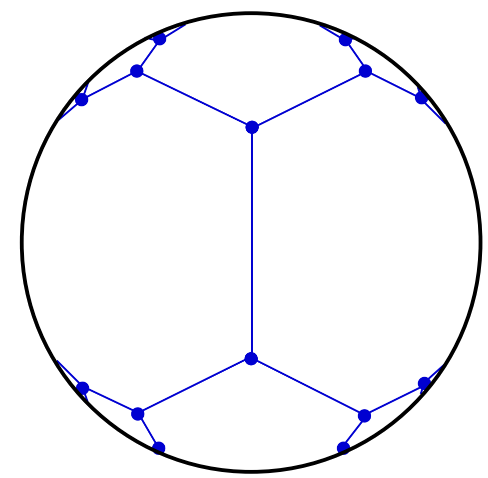

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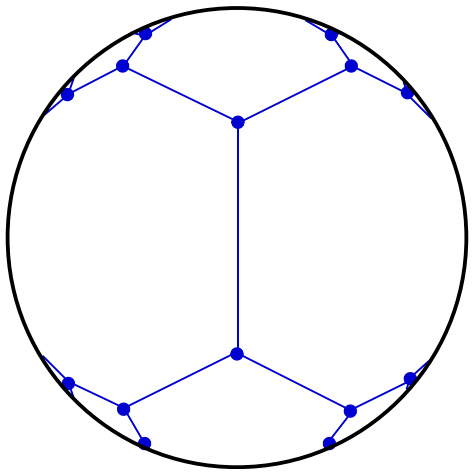
Can in particular be applied to obtain reduction domains for the action of $\mathrm{GL}_n(\mathcal{O}_K)$ on suitable quadratic spaces

Applications in Math

Ryshkov Polyhedron



$\mathrm{GL}_n(\mathcal{O}_K)$ symmetric

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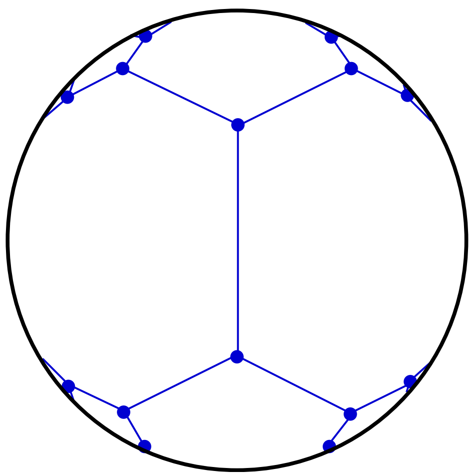

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Representation Conversion

Vertices / Perfect Forms:

- Reduction theory
- Hermite constant

Polyhedral complex:

- Cohomology of $\mathrm{GL}_n(\mathcal{O}_K)$
- Hecke operators
- Compactifications of moduli spaces of Abelian varieties

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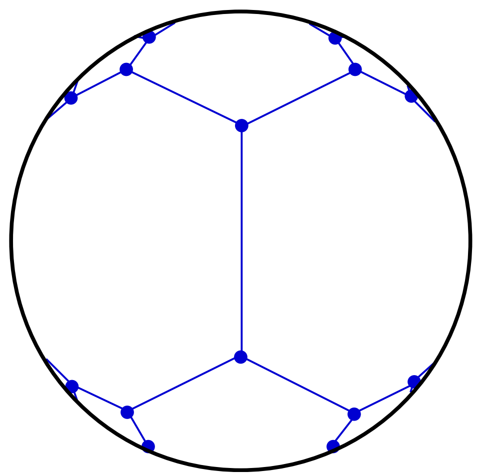

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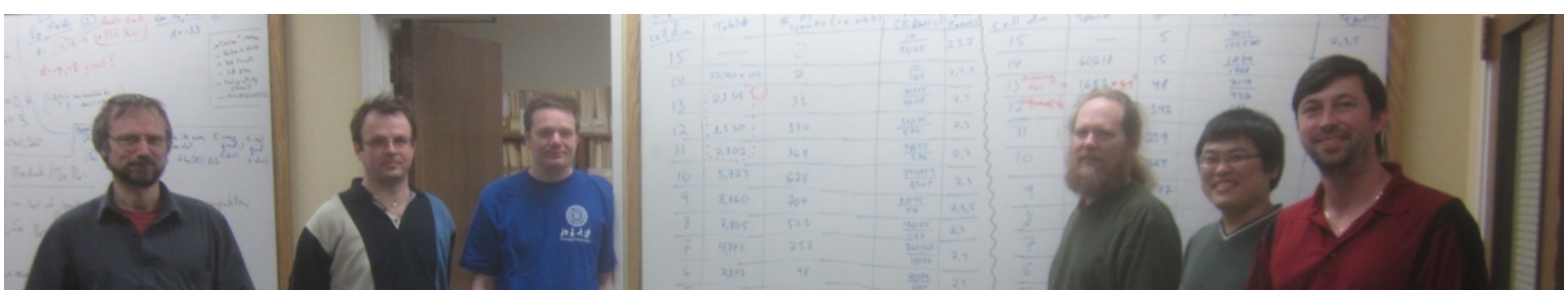
See Mathieu's talk after the coffee break!



AIM Square group 2012: Gangl, Dutour Sikirić, Schürmann, Gunnells, Yasaki, Hanke

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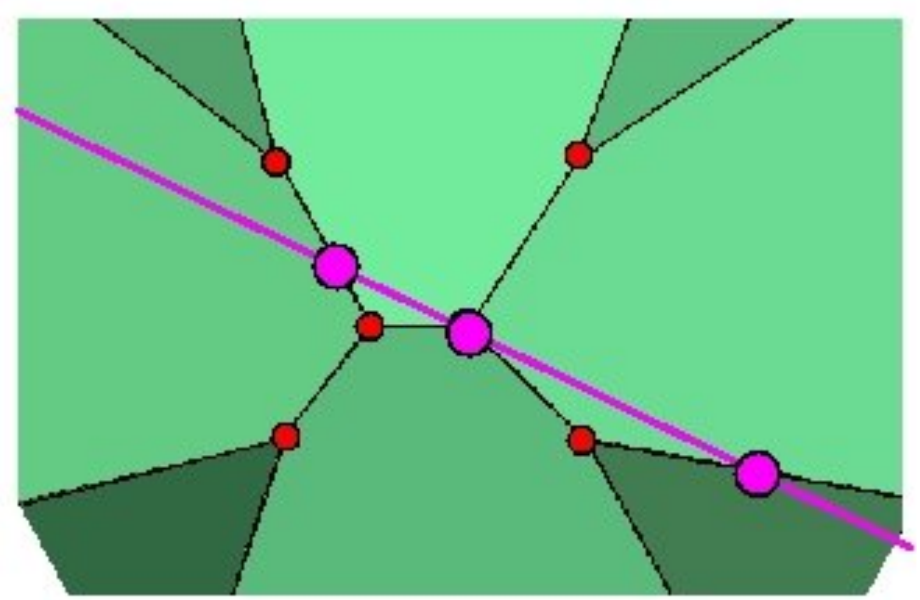
IDEA (Bergé, Martinet, Sigrist, 1992):

Intersect Ryshkov polyhedron $\mathcal{R}$ with a linear subspace $T \subset \mathcal{S}^n$



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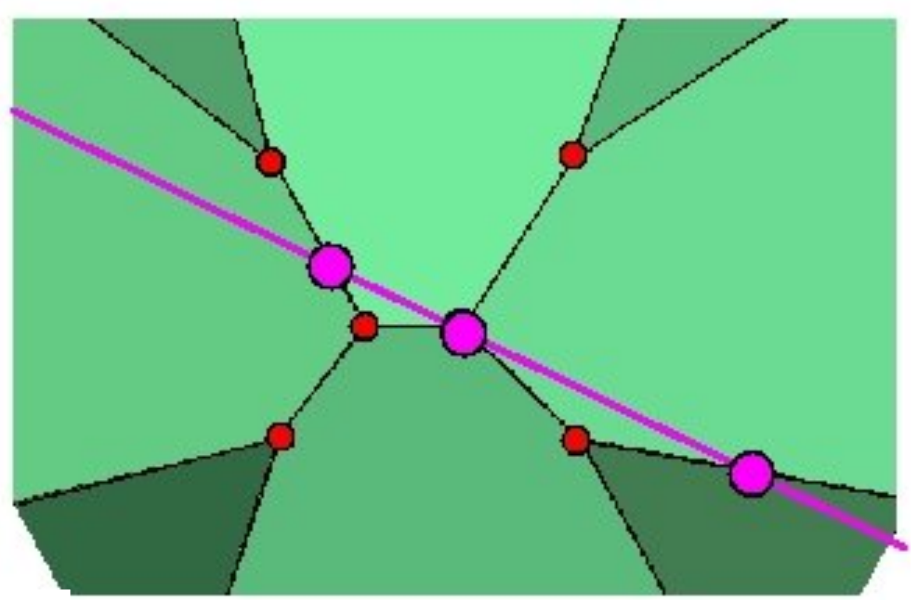
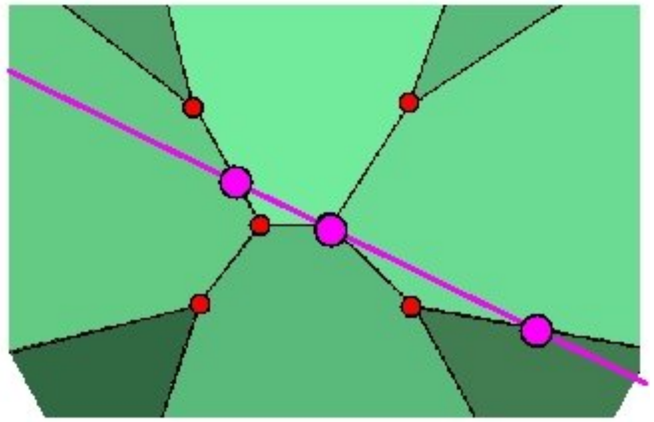


DEF: $Q \in T \cap \mathcal{S}^n_{>0}$ is T-perfect if it is a vertex of $\mathcal{R} \cap T$

Voronoi's Algorithm

for a linear subspace *T*



SVPs: Obtain a T-perfect form Q

1. SVP: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\; Q' \in T \;\;:\;\; Q'[x] \geq 1 \text{ for all } x \in \operatorname{Min} Q \;\}$$

2. PolyRepConv: Enumerate extreme rays $R_1, \ldots, R_k$ of $\mathcal{P}(Q)$
3. For the indefinite R_i, $i = 1, \ldots, k$

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4. T-ISOMs: Test if Q_i is T-equivalent to a known form
5. Repeat steps 1.–4. for new perfect forms

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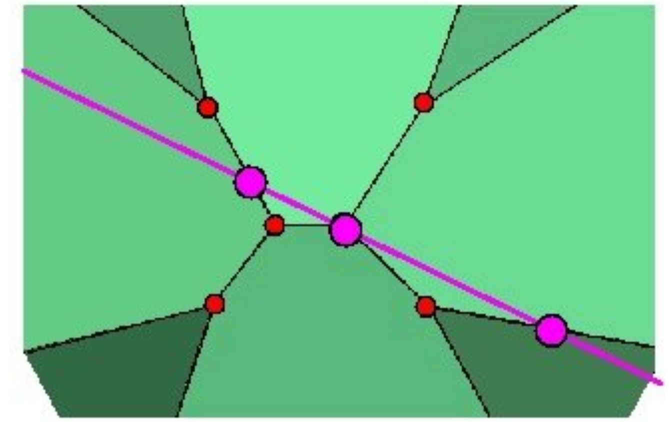

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Possible existence of "Dead-Ends" (for PQFs *R*)

G-invariant theory

$Q, Q' \in T \cap \mathcal{S}^n_{>0}$ are called T-equivalent, if $\exists U \in \mathrm{GL}_n(\mathbb{Z})$ with

$$Q' = U^t Q U \quad \text{and} \quad T = U^t T U$$

For a finite group $G \subset \mathrm{GL}_n(\mathbb{Z})$ the space of invariant forms

$$T_G \quad := \quad \{ \, Q \in \mathcal{S}^n \ : \ G \subset \operatorname{Aut} Q \, \}$$

is a linear subspace of $\mathcal{S}^n$; $\qquad T_G \cap \mathcal{S}^n_{>0}$ is called Bravais space

THM (Jaquet-Chiffelle, 1995):

$$\{ \, T_G\text{-perfect } Q \ : \ \lambda(Q) = 1 \, \} \ / \sim_{T_G} \quad \text{finite}$$

Applicaton: Lattice Sphere Packings

with prescribed symmetry

n	2	4	6	8	10	12
# $\mathcal{E}$-perfect	1	1	2	5	1628	?
maximum δ	$0.9069\ldots$	$0.6168\ldots$	$0.3729\ldots$	$0.2536\ldots$	$0.0360\ldots$	

Perfect Eisenstein forms

n	2	4	6	8	10	12
# $\mathcal{G}$-perfect	1	1	1	2	≥ 8192	?
maximum δ	$0.7853\ldots$	$0.6168\ldots$	$0.3229\ldots$	$0.2536\ldots$		

Perfect Gaussian forms

n	4	8	12	16
# $\mathcal{Q}$-perfect	1	1	8	?
maximum δ	$0.6168\ldots$	$0.2536\ldots$	$0.03125\ldots$	

Perfect Quaternion forms

PART III:
A new Generalization

Further Generalization? … and application!

IDEA: Generalize Voronoi's theory to
other convex cones and their duals

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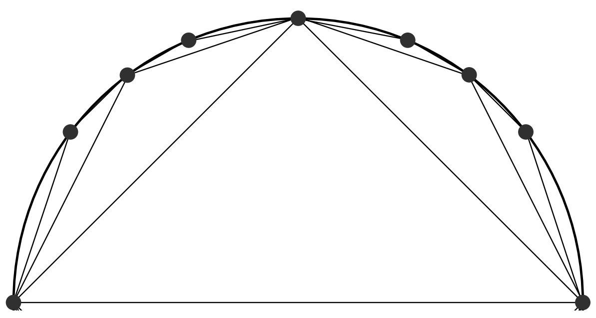

In particular to the completely positive cone

Further Generalization? … and application!

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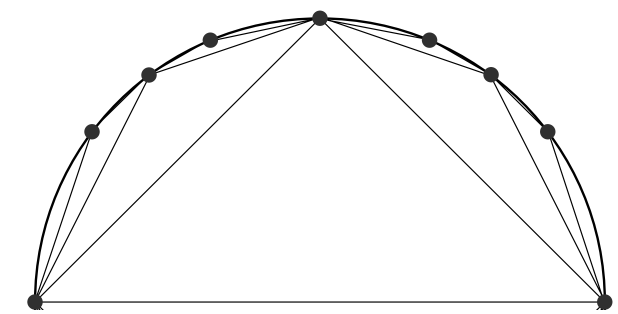


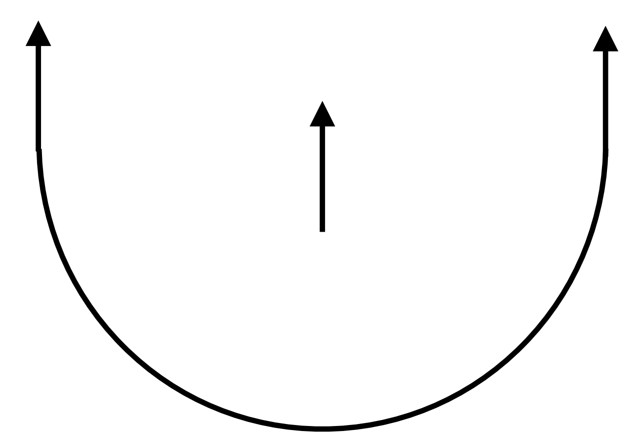
In particular to the completely positive cone

$$\mathcal{CP}_n = \mathrm{cone}\{xx^{\mathsf{T}} : x \in \mathbb{R}^n_{\geq 0}\}$$ and its dual, the copositive cone

$$\mathcal{COP}_n = (\mathcal{CP}_n)^* = \{B \in \mathcal{S}^n : \langle A, B \rangle \geq 0 \text{ for all } A \in \mathcal{CP}_n\}$$

$$= \{B \in \mathcal{S}^n : B[x] \geq 0 \text{ for all } x \in \mathbb{R}^n_{\geq 0}\}$$

$$\mathcal{CP}_n \quad \subset \quad \mathcal{S}^n_{>0} \quad \subset \quad \mathcal{COP}_n$$



$\langle A, B \rangle = \mathrm{Trace}(A \cdot B)$ denotes the standard inner product on $\mathcal{S}^n$

Application: Copositive Optimization

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- Copositive optimization problems are convex conic problems

$$\min \langle C, Q \rangle \quad \text{such that } \langle Q, A_i \rangle = b_i,\ i = 1, \ldots, m$$

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Such problems have a duality theory and allow certificates for solutions!

cp-factorizations and certificates

DEF: A finite set $X \subset \mathbb{R}^n_{\geq 0}$ is called a certificate for $Q \in \mathcal{S}^n$ being completely positive, if it gives a cp-factorization $Q = \sum_{x \in X} x x^\top$

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Non of these approaches is exact and latter do not even guarantee to find solutions!

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For $Q \in \operatorname{int} \mathcal{COP}_n$ we can construct a family of simplices Δ^k in the standard simplex $\Delta = \left\{x \in \mathbb{R}^n_{\geq 0} : x_1 + \ldots x_n = 1\right\}$ such that each Δ^k has vertices $v_1, \ldots v_n$ with $v_i^\top Q v_j > 0$

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Computation in practice:

"Fincke-Pohst strategy" to compute $\min_{\mathcal{COP}} Q$ in each cone Δ^k

Generalized Ryshkov polyhedron

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The set of all copositive quadratic forms / matrices with copositive minimum at least 1 is called Ryshkov polyhedron

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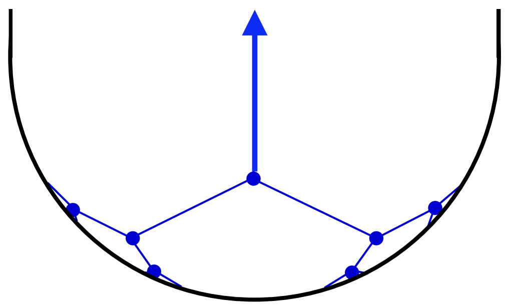

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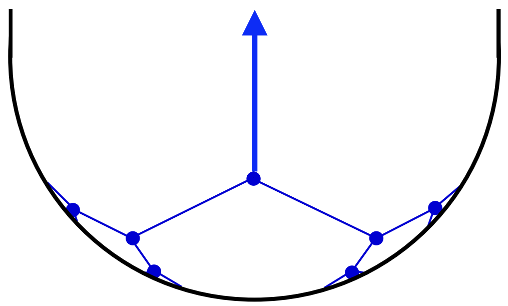

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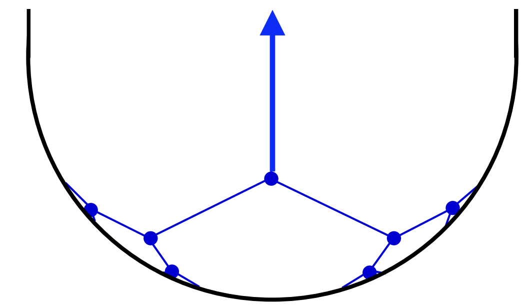

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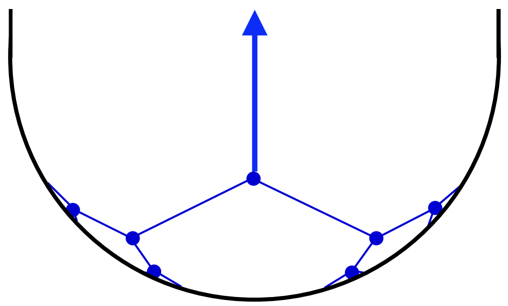

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2. LP: $\texttt{if}$ $A \in \text{cone}\left\{xx^\top : x \in \text{Min}_{\mathcal{COP}} B_P\right\}$ $\texttt{then}$ $\texttt{output}$ $A \in \tilde{\mathcal{CP}}_n$
3. COP-SVP: Compute $\text{Min}_{\mathcal{COP}} B_P$ and the polyhedral cone

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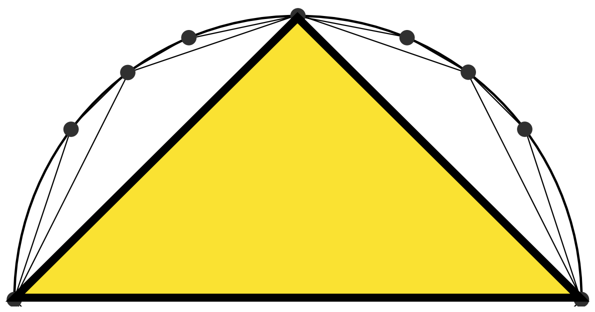

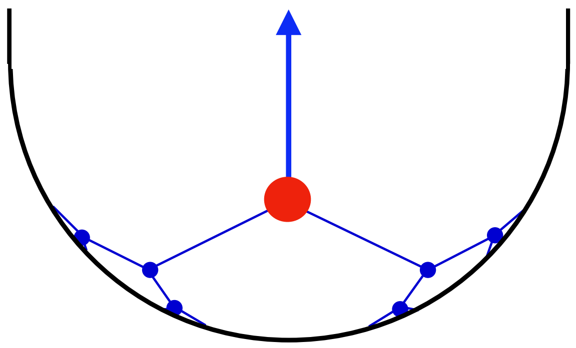

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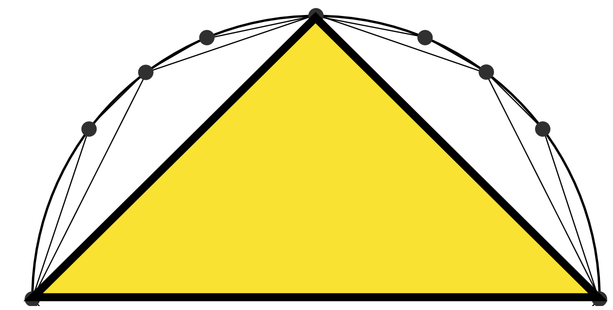

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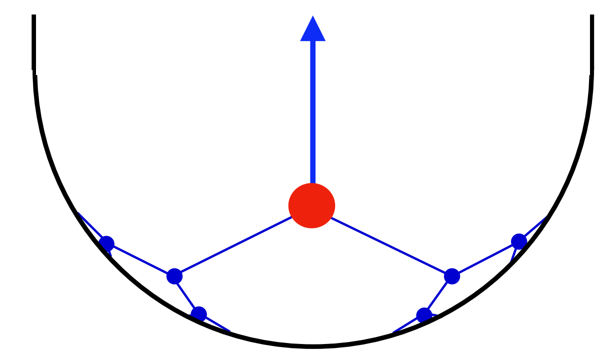

Proof. Matrix Q_{A_n} is positive definite since

$$Q_{\mathsf{A}_n}[x] = x_1^2 + \sum_{i=1}^{n-1}(x_i - x_{i+1})^2 + x_n^2 \qquad \text{for } x \in \mathbb{R}.$$

In particular it lies in the interior of the copositive cone. Furthermore,

$$\min_{\mathcal{COP}} Q_{\mathsf{A}_n} = 2 \quad \text{with} \quad \mathrm{Min}_{\mathcal{COP}} Q_{\mathsf{A}_n} = \left\{ \sum_{i=j}^{k} e_j : 1 \leq j \leq k \leq n \right\}$$

Interior cases

(algorithm terminates)

EX: $$A = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$$

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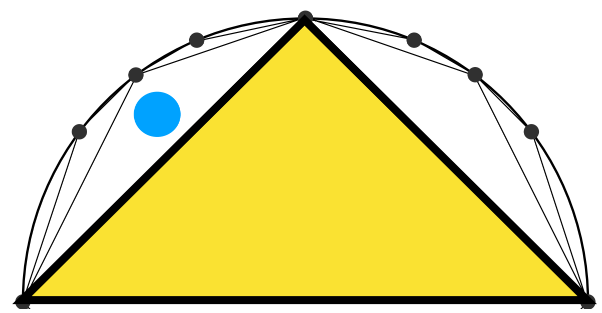

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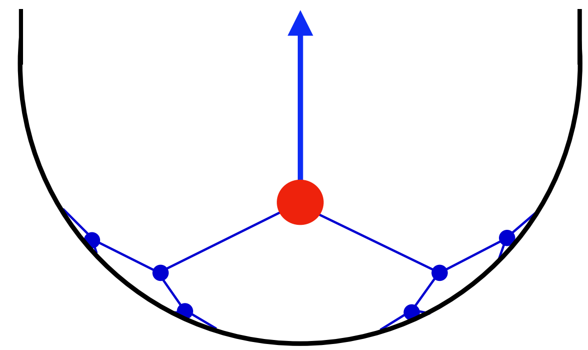

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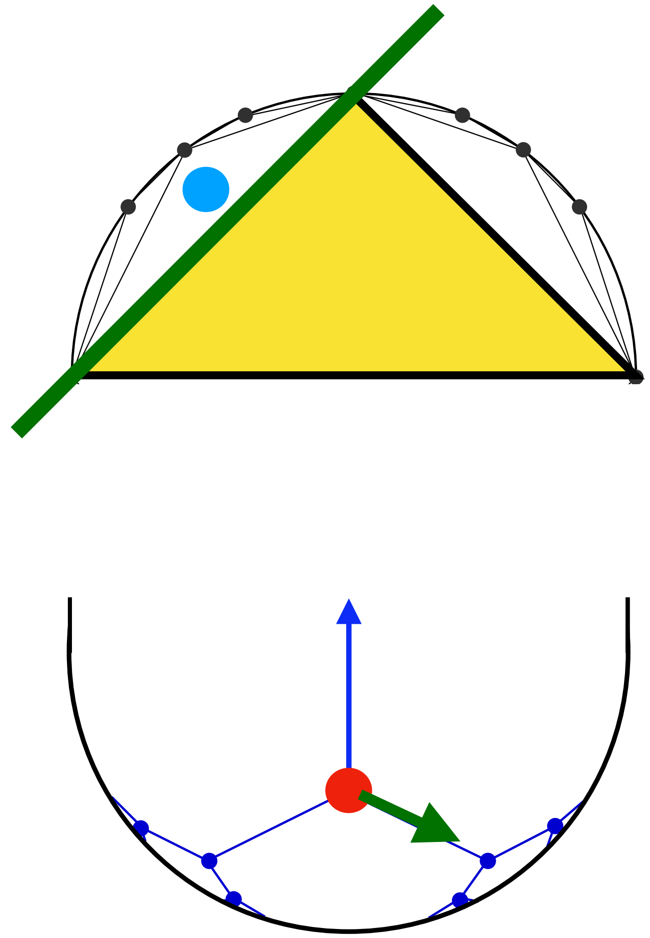

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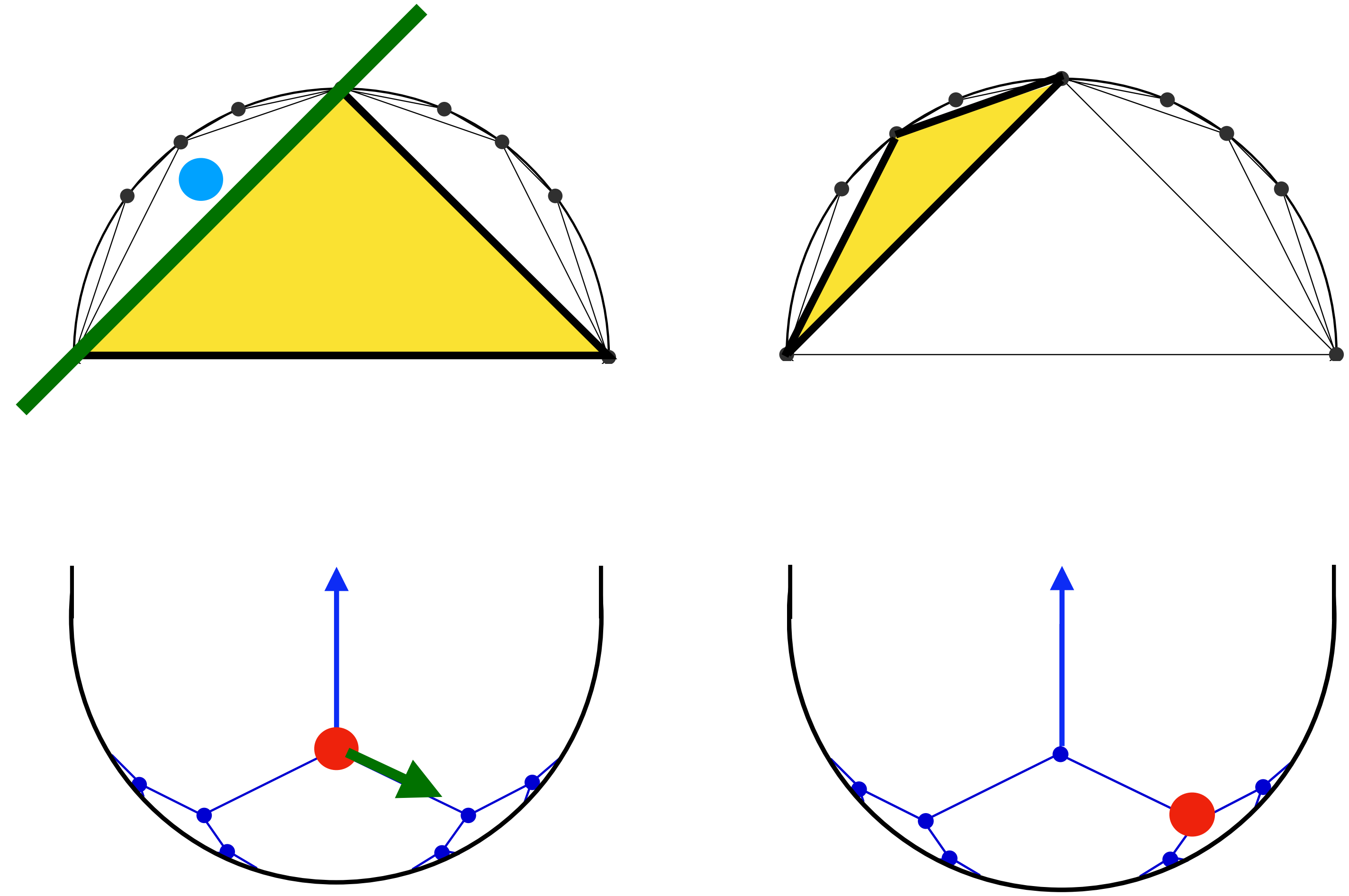

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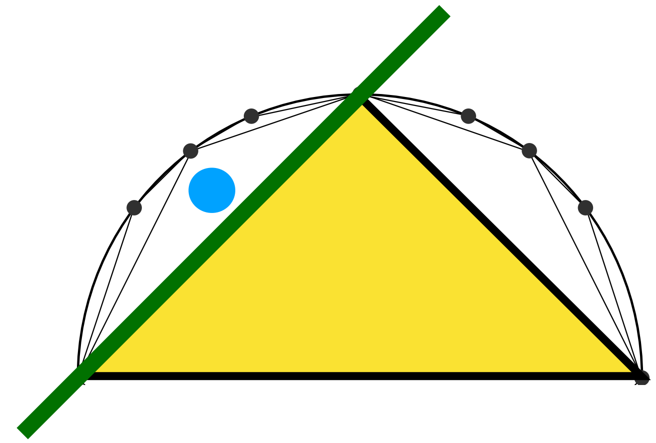

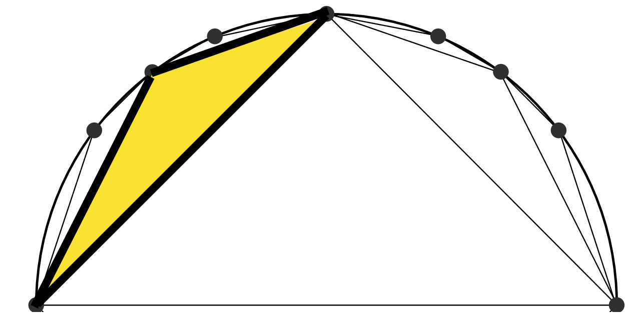

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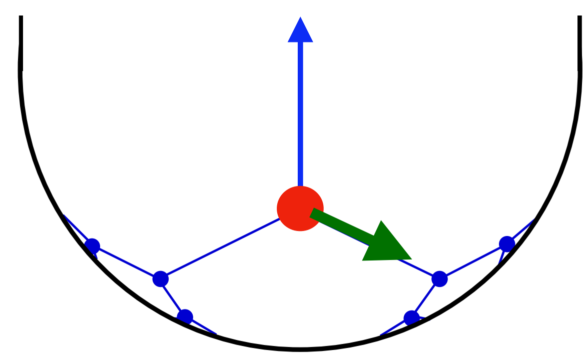
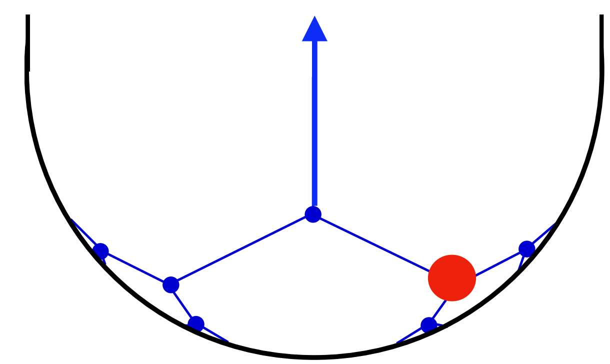


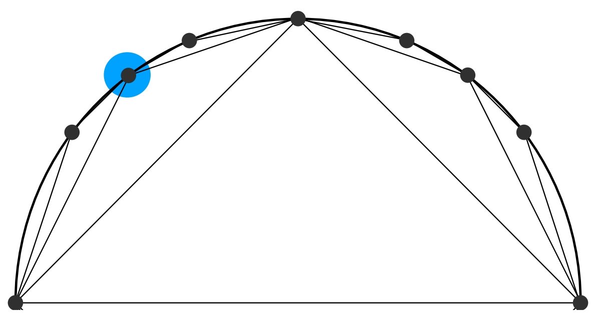


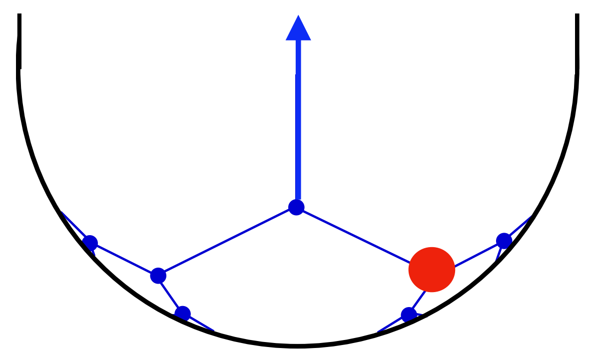
Starting with Q_{A_2} one iteration of the algorithm finds the $\mathcal{COP}$-perfect matrix $B_P = \begin{pmatrix} 1 & -3/2 \\ -3/2 & 3 \end{pmatrix}$ and

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\top + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}^\top$$

Boundary cases from $\tilde{\mathcal{CP}}_n$

(algorithm terminates with a suitable pivot-rule)





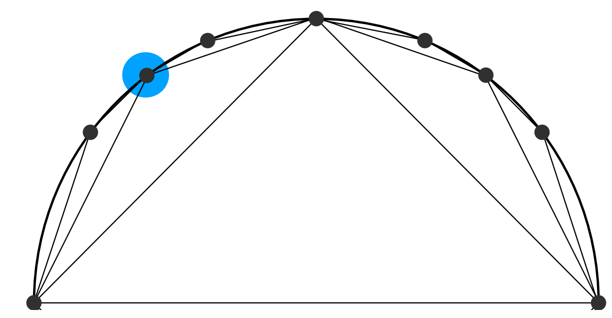
EX: $\begin{pmatrix} 8 & 5 & 1 & 1 & 5 \\ 5 & 8 & 5 & 1 & 1 \\ 1 & 5 & 8 & 5 & 1 \\ 1 & 1 & 5 & 8 & 5 \\ 5 & 1 & 1 & 5 & 8 \end{pmatrix}$

from Groetzner, Dür (2018)

not solved by their algorithms

Boundary cases from $\mathcal{C}\tilde{\mathcal{P}}_n$

(algorithm terminates with a suitable pivot-rule)



EX: $\begin{pmatrix} 8 & 5 & 1 & 1 & 5 \\ 5 & 8 & 5 & 1 & 1 \\ 1 & 5 & 8 & 5 & 1 \\ 1 & 1 & 5 & 8 & 5 \\ 5 & 1 & 1 & 5 & 8 \end{pmatrix}$

from Groetzner, Dür (2018)

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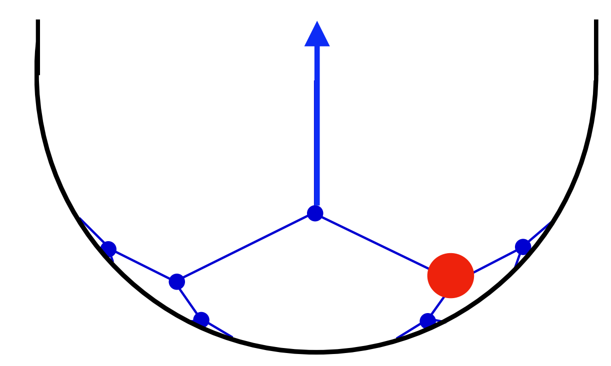


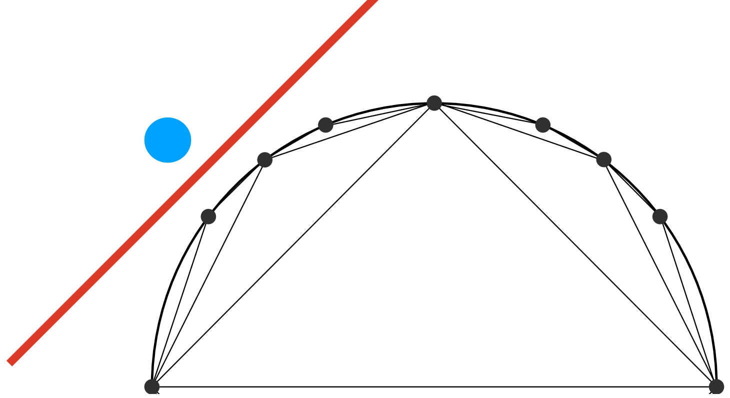
Starting with Q_{A_5}, our algorithm finds a cp-factorization after 5 iterations

$$\begin{aligned} v_1 &= (0,0,0,1,1) & v_6 &= (1,0,0,0,1) \\ v_2 &= (0,0,1,1,0) & v_7 &= (1,0,0,1,2) \\ v_3 &= (0,0,1,2,1) & v_8 &= (1,1,0,0,0) \\ v_4 &= (0,1,1,0,0) & v_9 &= (1,2,1,0,0) \\ v_5 &= (0,1,2,1,0) & v_{10} &= (2,1,0,0,1) \end{aligned}$$

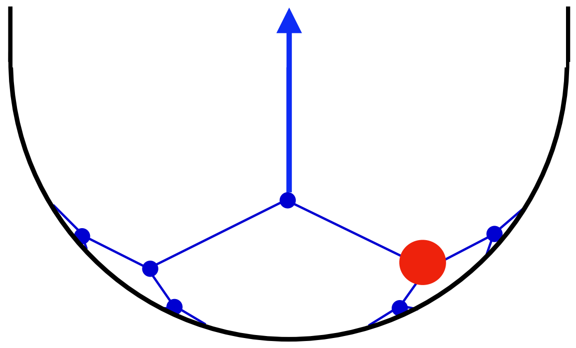
giving a certificate for the matrix to be completely positive

Exterior cases

(algorithm conjectured to terminate)

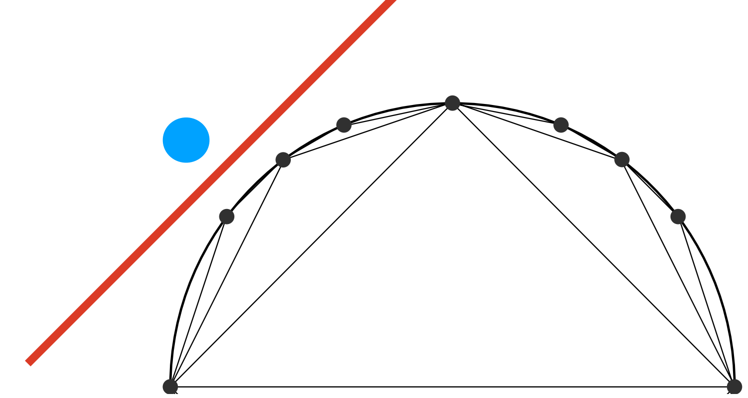


EX: $$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 6 \end{pmatrix}$$ from Nie (2014)

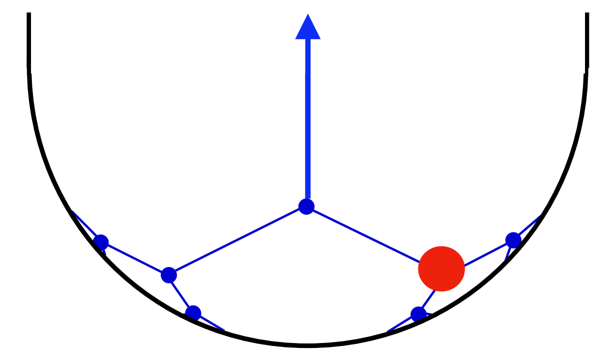


Exterior cases

(algorithm conjectured to terminate)



EX: $\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 6 \end{pmatrix}$ from Nie (2014)



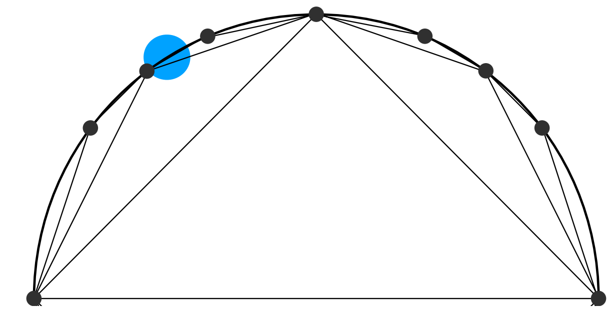
Starting with Q_{A_5}, after 18 iterations our algorithm finds the $\mathcal{COP}$-perfect

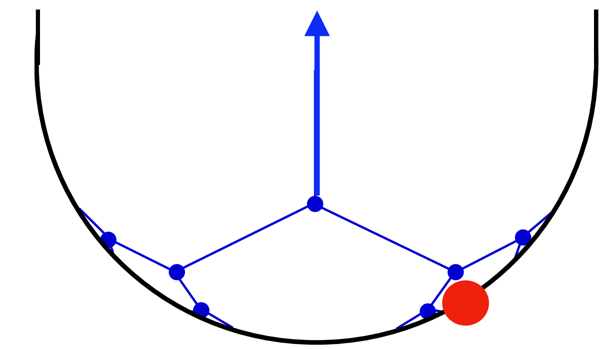
$$\begin{pmatrix} 363/5 & -2126/35 & 2879/70 & 608/21 & -4519/210 \\ -2126/35 & 1787/35 & -347/10 & 1025/42 & 253/14 \\ 2879/70 & -347/10 & 829/35 & -1748/105 & 371/30 \\ 608/21 & 1025/42 & -1748/105 & 1237/105 & -601/70 \\ -4519/210 & 253/14 & 371/30 & -601/70 & 671/105 \end{pmatrix}$$

giving a certificate for the matrix not to be completely positive

Irrational boundary cases

(algorithm is known not to terminate)

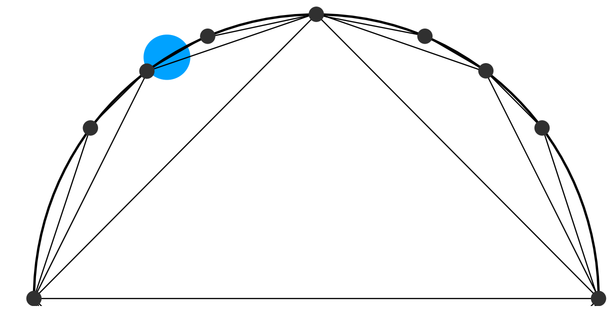




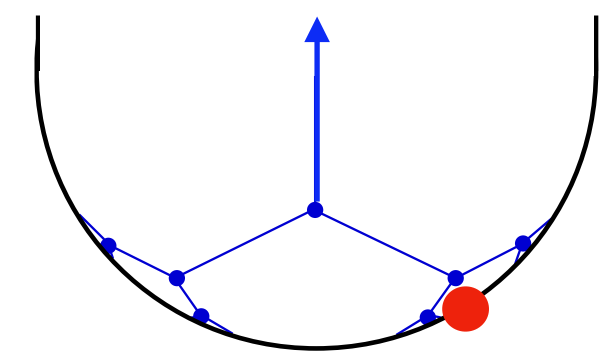
EX: $A = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}^{\top} = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$

Irrational boundary cases

(algorithm is known not to terminate)



EX: $A = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}^{\top} = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$



The $\mathcal{COP}$-perfect matrix after ten iterations of the algorithm is

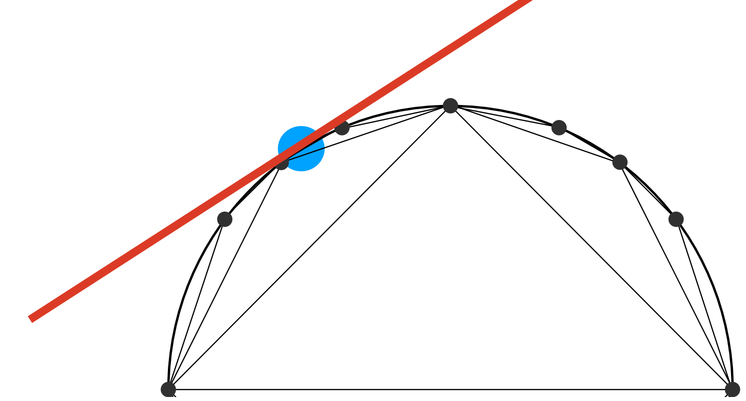
$$B_P^{(10)} = \begin{pmatrix} 4756 & -6726 \\ -6726 & 9512 \end{pmatrix}.$$

It can be shown that the matrices $B_P^{(i)}$ converge to a multiple of

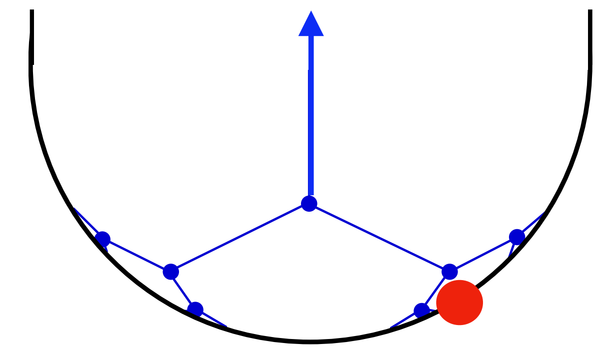
$$B = \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix} \text{ satisfying } \langle A, B \rangle = 0 \text{ and } \langle X, B \rangle \geq 0 \text{ for all } X \in \mathcal{CP}_2.$$

Irrational boundary cases

(algorithm is known not to terminate)



EX: $A = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}^{\top} = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$



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